Physics 517
Fall 2004
Due: 6 October 2004

Assignment # 5

1. The interval for Minkowski space can be written

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

(We choose units where c, the speed of light, is unity, $c = 1$.) The Lorentz transformations are important in relativity and E&M because they leave the form of the interval invariant. Consider a frame $\mathcal{O}$ with coordinates $\{t, x, y, z\}$, and a frame $\bar{\mathcal{O}}$ with coordinates $\{\bar{t}, \bar{x}, \bar{y}, \bar{z}\}$, moving with velocity v along the x-axis with respect to frame $\mathcal{O}$. The Lorentz transformation between these two frames can be written

$$\bar{t} = \gamma(t - vx), \tag{2}$$
$$\bar{x} = \gamma(x - vt), \tag{3}$$
$$\bar{y} = y, \tag{4}$$
$$\bar{z} = z, \tag{5}$$

where v is the boost velocity along the x-axis and $\gamma = (1 - v^2)^{-1/2}$.

(a) Find a matrix representation of the transformation $\Lambda^{\bar{a}}{}_b$ and its inverse, $\Lambda^a{}_{\bar{b}}$.

(b) Write the metric g_{ab} in matrix form.

(c) Transform the metric to obtain $g_{\bar{a}\bar{b}}$, and write the solution in matrix form.

(d) The Riemann curvature tensor is often written $R^a{}_{bcd}$. Write an expression for $R^{\bar{a}}{}_{\bar{b}\bar{c}\bar{d}}$ in terms of $R^a{}_{bcd}$ and the transformations matrices $\Lambda^{\bar{a}}{}_b$ and $\Lambda^a{}_{\bar{b}}$. Do not evaluate this expression.

2. $\mathbf{v}$ and $\mathbf{w}$ are vectors in the vector space, $\mathcal{V}$, with an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^N$. The dual basis in $\mathcal{V}^*$ is $\{\boldsymbol{\omega}^i\}_{i=1}^N$, and $\boldsymbol{\omega}^i(\mathbf{e}_j) = \delta^i{}_j$. A symmetric tensor $\mathbf{T}$ is constructed from $\mathbf{v}$ and $\mathbf{w}$ as $\mathbf{T} = \frac{1}{2}(\mathbf{v} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{v})(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2)$, where $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2$ are elements of $\mathcal{V}^*$.

(a) Expand $\mathbf{v}$ and $\mathbf{w}$ in the basis $\mathbf{e}_i$, and find the components of $\mathbf{T}$ in this basis, T^{ab}, in terms of v^i and w^j. The components are found by letting $\mathbf{T}$ act on the basis forms, $\boldsymbol{\omega}^i$: $T^{ab} = \mathbf{T}(\boldsymbol{\omega}^a, \boldsymbol{\omega}^b)$. Show that the contravariant components, T^{ab}, are symmetric, $T^{ab} = T^{ba}$.

(b) Show that the covariant components, T_{ab}, are symmetric, $T_{ab} = T_{ba}$. Begin with the result of part (a), and assume the existance of a metric, g_{ab}, that may be used to lower indices.

(c) $\mathbf{A}$ is an anti-symmetric tensor, $A_{ab} = -A_{ba}$. Show that $A_{ab}T^{ab} = 0$.

3. $\mathbf{A}$ is a generic rank (0,2) tensor with components A_{ab}.

   (a) Write an expression for the completely anti-symmetric tensor $A_{[ab]}$ in terms of the components A_{ab}.

   (b) Write an expression for the completely symmetric tensor $A_{(ab)}$ in terms of the components A_{ab}.

   (c) Show that $\mathbf{A}$ can be written in terms of $A_{(ab)}$ and $A_{[ab]}$.

   (d) $\mathbf{C}$ is a symmetric tensor, $C^{ij} = C^{(ij)}$. Show that $A_{ij}C^{ij} = A_{(ij)}C^{ij}$.

   (e) $\mathbf{D}$ is an anti-symmetric tensor, $D^{ij} = D^{[ij]}$. Show that $A_{ij}D^{ij} = A_{[ij]}D^{ij}$.

4. $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, are elements of a 4-dimensional vector space, $\mathcal{V}$, with an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^{4}$:

$$\mathbf{v}_1 = \mathbf{e}_1 + 3\mathbf{e}_2 - 2\mathbf{e}_3 + \mathbf{e}_4 \quad (6)$$
$$\mathbf{v}_2 = -\mathbf{e}_2 + 2\mathbf{e}_4 \quad (7)$$
$$\mathbf{v}_3 = \mathbf{e}_1 + 5\mathbf{e}_3 - 3\mathbf{e}_4 \quad (8)$$
$$\mathbf{v}_4 = -\mathbf{e}_1 - 12\mathbf{e}_3 + 13\mathbf{e}_4 \quad (9)$$

   Are these vectors linearly independent? Answer this question by calculating the exterior product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4$. If the vectors are linearly independent, then they could form a basis in $\mathcal{V}$, and the exterior product must be non-zero. (Hint: Consider first calculating $\mathbf{v}_1 \wedge \mathbf{v}_2$ and $\mathbf{v}_3 \wedge \mathbf{v}_4$ individually, before calculating the final product. Why does this simplify the calculation?)

5. Show that when there is a sum over an upper index and a lower index, swapping the upper index to a lower index, and vice versa, does not change the sum. In other words, $A^a B_a = A_a B^a$.

6. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an oriented, orthonormal basis in three dimensions. Evaluate these expressions with the Hodge star operator:

   (a) $*1$

   (b) $*\mathbf{e}_1$

   (c) $*(\mathbf{e}_1 \wedge \mathbf{e}_3)$

   (d) $*(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$

7. The line element for a black hole spacetime can be written

$$ds^2 = -A^2\,dt^2 + 2B\,dtdr + C^2\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \qquad (10)$$

where A, B and C are functions, and the coordinates are $\{t, r, \theta, \phi\}$.

(a) Find the metric components g_{ab}. Write the metric in matrix form.

(b) The four-velocity of a particle in this spacetime is $u^a = (1, 0, 0, 0)$. Find the components of u_a.

8. $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 3-dimensional Euclidian space. Show that

(a) $*(\mathbf{u} \wedge \mathbf{v})$ is equivalent to $\mathbf{u} \times \mathbf{v}$;

(b) $*((*\mathbf{u}) \wedge \mathbf{v})$ is equivalent to $\mathbf{u} \cdot \mathbf{v}$;

(c) $*(\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v})$ is equivalent to $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$;

(d) $*(\mathbf{w} \wedge *(\mathbf{u} \wedge \mathbf{v}))$ is equivalent to $\mathbf{w} \times (\mathbf{u} \times \mathbf{v})$.

9. Let $\mathbf{u}$ be an unit vector, $\mathbf{u} \cdot \mathbf{u} = 1$. A projection operator, $\mathbf{P_u}$, may be defined by

$$\mathbf{P_u} = \mathbf{g} - \mathbf{u} \otimes \mathbf{u} \qquad (11)$$

where $\mathbf{g}$ is the metric tensor. $\mathbf{P_u}$ projects an arbitrary vector $\mathbf{v}$ into $\mathbf{v}_\perp$, which is orthogonal to $\mathbf{u}$: $\mathbf{v}_\perp = \mathbf{P_u}(\mathbf{v})$ and $\mathbf{v}_\perp \cdot \mathbf{u} = 0$. In component form, the projection can be written

$$v^a_\perp = P^a{}_b v^b = (g^a{}_b - u^a u_b) v^b \qquad (12)$$

(a) Show explicitly that $\mathbf{v}_\perp \cdot \mathbf{u} = 0$.

(b) Show that multiple applications of $\mathbf{P_u}$ do not change the vector $\mathbf{v}_\perp$, i.e., $\mathbf{P_u}(\mathbf{v}_\perp) = \mathbf{v}_\perp$.

(c) Show that $\mathbf{P_u}$ is the metric tensor for vectors perpendicular to $\mathbf{u}$: $\mathbf{P_u}(\mathbf{v}_\perp, \mathbf{w}_\perp) = \mathbf{g}(\mathbf{v}_\perp, \mathbf{w}_\perp) = \mathbf{v}_\perp \cdot \mathbf{w}_\perp$. (Write $\mathbf{P_u}$ with both indices down.)

10. **Either** work problem 2.5.10 from Arfken, **or** explain in two sentences (or fewer) why you would rather not work problem 2.5.10 from Arfken.