Physics 602
Winter 2008
Due: 25 February 2008

Assignment # 4

1. The rate of a particular chemical reaction $A + B \to C$ is proportional to the concentrations of the reactants A and B:

$$\frac{dC}{dt} = \alpha\,[A(0) - C(t)]\,[B(0) - C(t)]$$

Let $C(0) = 0$.

   (a) Find $C(t)$ for $A(0) \neq B(0)$.
   (b) Find $C(t)$ for $A(0) = B(0)$.

2. A boat moving initially with constant velocity experiences a resisting force proportional to v^n

$$m\frac{dv}{dt} = -kv^n,$$

where v is the velocity. Find $v(t)$ and the position, $x(t)$, with the initial conditions $v(0) = v_0$ and $x(0) = 0$.

3. Bernoulli's equation

$$\frac{dy}{dx} + f(x)y = g(x)y^n$$

is nonlinear for $n > 1$. Show that the substitution $u = y^{1-n}$ reduces this equation to a linear equation.

4. Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that a second linearly independent solution is $R = r^{-m}$.

5. Find the general solution of the differential equation

$$\frac{d^3u}{dt^3} + 6\frac{d^2u}{dt^2} + 12\frac{du}{dt} + 8u = 12e^{-t}$$

6. Find the general solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{1+x^2} \qquad -\infty < x < \infty$$

7. Consider the differential equation

$$\frac{d^2y}{dt^2} + y(t) = g(t)$$

   (a) For the initial value problem

$$y(0) = 0, \quad y'(0) = 0.$$

   Show that

$$y = \phi(t) = \int_0^t g(s)\sin(t-s)\,ds.$$

   This formula shows the relation between the input, $g(t)$, and the output $\phi(t)$. The output at time t depends on the input from the initial time, $t = 0$, through time t. This integral is the *convolution* of $\sin(t)$ and $g(t)$.

(b) Now let

$$y(0) = y_0, \qquad y'(0) = y_0'$$

and show that

$$y(t) = y_0 \cos t + y_0' \sin t + \int_0^t g(s) \sin(t-s)\, ds.$$

8. Solve the system

$$\ddot{x} = -2\dot{x} - 5y + 3$$
$$\dot{y} = \dot{x} + 2y$$

with the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, and $y(0) = 1$. *Hint.* Write these equations as a system of first order equations.

9. Consider the linear system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}(t).$$

(a) Show that $\lambda = 1$ is a triple eigenvalue of $\mathbf{A}$, and that there are only two linearly independent eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Find two linearly independent solutions, $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$.

(b) To find the third solution, assume that

$$\mathbf{x}^{(3)}(t) = \boldsymbol{\xi}\, te^t + \boldsymbol{\eta}\, e^t$$

and show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$
$$(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

(c) Show that $\boldsymbol{\xi} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$, where c_1 and c_2 are arbitrary constants, is the most general solution of $(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = 0$. Show that in order to solve for $\boldsymbol{\eta}$ it is necessary that $c_1 = c_2$.

(d) Obtain $\mathbf{x}^{(3)}(t)$.