Waveguide Guide: **A** and V

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I really think that waveguide fields are easier to understand using the potentials **A** and V than they are using the electric and magnetic fields. Since Griffiths doesn't do it this way, I will have to show you what I mean. The starting point is Maxwell's equations for the potentials, which are just the wave equations (in the Lorentz gauge).

Wave Equations:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \quad ; \quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{1}$$

Lorentz Condition:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \tag{2}$$

And, of course, when we need to find **E** and **B** that's easy too.

E and **B**:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad ; \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

To find out how the fields behave in a waveguide we use not only the wave equations, but also the boundary conditions.

Boundary Conditions:

$$\mathbf{E}_{\parallel} = 0 \quad ; \quad \mathbf{B}_{\perp} = 0 \tag{4}$$

Even though the discussion in the book about waveguides looks complicated, it really isn't; the electromagnetic fields inside waveguides are just superpositions of waves. They are traveling waves parallel to the axis of the guide and standing waves transverse to it. In a rectangular guide the fields are just superpositions of fields that are proportional to the usual complex wave function

$$e^{i(k_x x + k_y y + k_z z - \omega t)} \tag{5}$$

and hence the dispersion relation is exactly the same as in infinite space:

Dispersion Relation in Rectangular Coordinates:

It is the same for traveling, standing, or combined standing/traveling (waveguide) waves:

$$\omega^2 = (k_x^2 + k_y^2 + k_z^2)c^2 \tag{6}$$

This dispersion relation comes from the wave equations, so now we have to worry about the boundary conditions. The boundary conditions impose restrictions on the values of the components of $\mathbf{k}$ transverse to the guide, so we need now to specify the geometry we will be working in.

Rectangular Waveguide:

The wave guide is infinitely long in the z-direction and goes from 0 to a in the x-direction and from 0 to b in the y-direction.

Since the x and y directions are the transverse ones, we use wave functions that are traveling waves in z and have standing wave forms in x and y.

Waveguide Waveform:

$$\mathbf{A}(z,x,y,t) = \mathbf{A}(x,y)e^{i(k_z z - \omega t)} \quad ; \quad V(z,x,y,t) = V(x,y)e^{i(k_z z - \omega t)} \quad \Rightarrow \quad \frac{\partial}{\partial z} = ik_z \quad \text{and} \quad \frac{\partial}{\partial t} = -i\omega \tag{7}$$

where $\mathbf{A}(x,y)$ and $V(x,y)$ will turn out to involve sines and cosines.

Applying the boundary conditions now and doing the algebra is rather complicated, so I will just tell you that it turns out that there are two distinct types of solutions. One type has $E_z = 0$, i.e., the mode electric field only has components perpendicular (transverse) to the long axis of the wave guide. These are the TE (Transverse Electric field only) modes. The second type has $B_z = 0$, i.e., the mode magnetic field only has components perpendicular (transverse) to the long axis of the wave guide. These are the TM (Transverse Magnetic field only) modes. If you work with $\mathbf{E}$ and $\mathbf{B}$, this minor miracle reduces the number of vector components you need to find from 6 to 5; not really a big deal. But if you work with $\mathbf{A}$ and V, as we are doing here, it reduces the number from 4 to 2, which is a big help.

Once we know that there is a separation into two mode types the details are not hard to work out.

1 TE Modes

TE_{mn} Modes ($A_z = 0$ and $V = 0$):

Transverse Electric means that $E_z = 0$, and since

$$E_z = -\frac{\partial A_z}{\partial t} - \frac{\partial V}{\partial z} = i\omega A_z - ikV \tag{8}$$

an easy way to get $E_z = 0$ is to take $A_z = 0$ and $V = 0$. This turns out to be exactly right. Hence, we only have to find $A_x(x, y)$ and $A_y(x, y)$. Applying both the boundary conditions and the Lorentz gauge condition leads to

$$k_x = \frac{m\pi}{a} \quad ; \quad k_y = \frac{n\pi}{b} \tag{9}$$

and

$$A_x = A_{x0} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ; \quad A_y = A_{y0} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{10}$$

with

$$A_{x0}\frac{m}{a} + A_{y0}\frac{n}{b} = 0 \tag{11}$$

This condition relates the two amplitudes, but does not determine the overall magnitude. This is determined by the person who shoots the energy into the waveguide.

TE$_{0n}$ and TE$_{m0}$ Special Cases:

If you set either $m = 0$ or $n = 0$ in the amplitude relation above you will see that only one of the vector field components survives.

$$m = 0 \quad \Rightarrow \quad k_x = 0 \quad ; \quad k_y = \frac{n\pi}{b} \quad ; \quad A_x = A_0 \sin\left(\frac{n\pi y}{b}\right) \quad ; \quad A_y = 0 \tag{12}$$

$$n = 0 \quad \Rightarrow \quad k_x = \frac{m\pi}{a} \quad ; \quad k_y = 0 \quad ; \quad A_x = 0 \quad ; \quad A_y = A_0 \sin\left(\frac{m\pi x}{a}\right) \tag{13}$$

2 TM Modes

TM$_{mn}$ Modes ($A_x = 0$ and $A_y = 0$):

Transverse Magnetic means that $B_z = 0$, and since

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{14}$$

an easy way to get $B_z = 0$ is to take $A_x = 0$ and $A_y = 0$. This turns out to be exactly right. Hence, we only have to find $A_z(x, y)$ and $V(x, y)$. Applying both the boundary conditions and the Lorentz gauge condition leads to

$$k_x = \frac{m\pi}{a} \quad ; \quad k_y = \frac{n\pi}{b} \tag{15}$$

$$A_z = A_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ; \quad V = \frac{k_z c^2}{\omega} A_z \tag{16}$$

TM$_{0n}$ and TM$_{m0}$ Special Cases:

Setting $m = 0$ or $n = 0$ in the formula for A_z makes the fields vanish, so there are no such modes. This means that $m \geq 1$ and $n \geq 1$ for TM modes.

3 TEM Modes

In a completely hollow waveguide waves with both $\mathbf{E} = 0$ and $\mathbf{B} = 0$ parallel to the axis of the guide are impossible. But with a conductor along the axis these waves are possible. Their dispersion relation is simply

$$\omega = kc \tag{17}$$

with $\mathbf{k}$ parallel to the axis of the guide. The electric field points outward from the central conductor and terminates on the outer surface of the guide while the magnetic field circulates around the central conductor and runs parallel to the outer conductor. Hence, they are just about like free space waves with $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$ mutually perpendicular. In a cylindrical guide (a coaxial cable) of radius a these waves are described in cylindrical coordinates by

$$\mathbf{k} = k\hat{z} \quad ; \quad E_r = E_0 \, \frac{ae^{i(kz-\omega t)}}{s} \quad ; \quad B_\theta = \frac{E_0}{c} \, \frac{ae^{i(kz-\omega t)}}{s} \tag{18}$$

where E_0 is the electric field amplitude at $s = a$ and where s is the radial coordinate in cylindrical coordinates.

4 Waveguide Problems

1. (Griffiths Problem 9.27) Show that the mode TE_{00} cannot occur in a rectangular waveguide by working with the equations for A_x and A_y given in the *Waveguide Guide*. (Note: this is a real problem the way Griffiths does waveguides, but here it is trivial.)

2. (Griffiths Problem 9.30) Using the wave equations for $\mathbf{A}$ and V, the Lorentz gauge condition, and the boundary conditions, derive the properties of TM modes in a rectangular waveguide, i.e., derive Eqs. (6), (15), and (16) in the *Waveguide Guide*. Just follow the procedure we followed in class for the TE modes. (You may take as given that $A_x = 0$ and $A_y = 0$ and that A_z and V are products of sines and/or cosines of the arguments $k_x x$ and $k_y y$.) In particular, show the following:

(a)

$$k_x = \frac{m\pi}{a} \quad ; \quad k_y = \frac{n\pi}{b}$$

and

$$\omega^2 = (k_x^2 + k_y^2 + k_z^2)c^2$$

(b)

$$A_z = A_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ; \quad V = \frac{k_z c^2}{\omega} A_z$$

(c) Verify the statement in the section on TM modes that TM_{0n} and TM_{m0} modes cannot exist.

(d) Make some kind of a rough 3-d sketch of the electric and magnetic fields of a TM_{12} mode in a rectangular waveguide. I think the best way to do this is to make sketches of both $\mathbf{E}$ and $\mathbf{B}$ in the xy-plane for the x and y components of the fields, and also a contour plot of E_z in the xy-plane. Then make a side-view sketch of the $\mathbf{E}$-lines in the xz plane. Maple and Matlab can help here.