**Course Syllabus for Physics 721**

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Office Hours: MWF 1-2    Office: N281A ESC
Text: Classical Mechanics, 3$^{\text{rd}}$ Edition (Goldstein, Poole, and Safko)

**Course objectives**

This course will teach you the concepts of advanced mechanics and challenge you to improve your problem-solving skills. While $\mathbf{F} = m\mathbf{a}$ may seem like all you need to know about mechanics, it plays no role in the microscopic world. Instead, the sophisticated ideas of Lagrangian and Hamiltonian dynamics seem to provide the best mathematical description of the microscopic world, and these will be our topic this semester.

Another objective of the course is to teach you to write like a scientist. To help you develop this skill some of the homework problems are to be fully written up using Maple of Mathematica.

**Reading**

This course is hard. It's hard for me, and it will be hard for you, so real effort will be required of all of us. I expect you to read the assigned sections of the text before class (see the course calendar), where "read" means that we have pencil and paper handy to work out things that we don't understand as we go along. Pay attention to the assigned sections–we will be skipping some portions of the book. Occasional reading quizzes will be given during the first 5 minutes of class. Unless otherwise indicated, these quizzes are to consist of a paragraph containing coherent English sentences.

Occasionally a homework assignment will also include a historical reading assignment about one, or more, of the famous names in the development of dynamics. Many books have been written about these figures, but reading the Wikipedia article will be sufficient for this assignment.
Possible subjects include Galileo, Newton, Bernoulli (many possibilities), D'Alembert, Euler, Lagrange, Laplace, Legendre, Coriolis, Gauss, Hamilton, Jacobi, Poisson, Liouville, Noether, Lorentz, Poincare, and Einstein.

**Writing**

• Homework and the Written Word

About 8 of the homework problems are printed in boldface type. This means that they are to be fully written up in scientific style using Maple or Mathematica, thinking of your fellow 721 students as your audience. Because they are homework problems these documents will mostly consist of a series of equations and other mathematical expressions that lead, step by step, to the final answer. Your job is to place coherent English sentences between these equations so that the flow from problem statement to solution is clear.

Each problem should begin with a centered title followed by your name (also centered). These are followed by a paragraph restating the problem and then another paragraph describing the approach you will take in solving it. As you proceed with the mathematics be sure to define all of the variables you use and indicate how each step of the solution leads to the next. Discuss anything interesting that shows up along the way and write a paragraph at the end indicating what has been accomplished. You will find an example of such a problem solution on the course web page.

When I grade these problems I will be looking at both the mathematics and the writing, equally weighted in the grading. To receive full credit the problem must be done correctly and the description of what has been accomplished must consist of complete sentences that flow well together. Correct grammar, word usage, and clarity of expression are also expected. These exercises will count for 5% of your grade in the course.

**Homework**

The purpose of the homework is to help you learn the material sufficiently well that you can pass the examinations. I encourage you to work together on these problems so that you can teach this material to each other, and I encourage you to come see me when you get stuck. We may also schedule a homework help session if the class feels that this would be helpful. But the homework problems that you hand in should be your own work, not the work of a committee. Even though you may have worked a problem out on the board as a group, each member of the group should write out his or her own version of the solution.

As you do the homework it is all right to do algebra and solve differential equations with Maple or Mathematica. Your performance on the homework will count for only 25% of your final grade (actually 30% counting the written-up problems described in the writing section above). This means that doing well on the homework and failing the exams will result in a failing grade for the course.

The assigned problems are at the end of this syllabus and their due dates are on the course calendar.

**Examinations and grading**

There will be three examinations during the semester, as noted on the course calendar. All will be timed in-class examinations. The final exam is scheduled for Wednesday, Dec. 17 from 11:00 AM-20:00 PM. Judging by my previous interactions with the scheduling office on the times of final exams, this time will not be changed.

Final grades will be based on the following weights: 5% for the homework writing assignments, 10% for the historical paper, 25% for the semester exams, 25% for the homework, 35% for the final.

**Assigned problems**

**Homework assignment 1:** Chapter 1: 1, 2, and 3. Read about Galileo and Newton.

**Problem A (Maple writing assignment)**: A particle of mass $m$ is falling under uniform gravity ($g$) with an air friction force of magnitude $bv^2$ pulling on it opposite to the direction of $v$ (with $b$ a constant). Assuming that it is dropped from rest with initial position $x = 0$, find formulas for $v(t)$, $x(t)$, and for the terminal speed. Make plots of $x(t)$ and $v(t)$ and discuss why the look as they do, both early and late in time.

**Homework assignment 2:** Chapter 1: 8, 10, 13, and 14. Read about D'Alembert and Lagrange. Chapter 2: 3

Problem B: Consider a frictionless block of mass $M$ which is free to slide along a track in the $x$ direction. Let it's $x$-displacement be $X(t)$. Hanging from the center of this mass is a pendulum of length $R$ and mass $m$ acted upon by gravity: $V = mgy$. Use the usual angle $\theta(t)$ to describe its motion. Write down the Lagrangian for this system, find any obvious constants of the motion, and use the standard formalism to find the second-order differential equation for $\theta(t)$. Then find the frequency of small amplitude vibrations about $\theta = 0$. Warning: The equations of motion for $X$ and $\theta$ are coupled. Be careful as you write down the kinetic energy for $m$.

**Problem C (Maple writing assignment):** Consider a sphere of mass $M$ and radius $R$ rolling down an inclined plane making angle $\alpha$ with the horizontal direction and with the end of the incline (the pointed end) sitting at the edge of a table. Using the horizontal position $x$ of the center of the sphere as the generalized coordinate, find and solve the equation of motion of of the sphere if it is released from rest at the top of the incline where $x = 0$ and where the height above the floor is $h$. Find a formula for the time when the sphere reaches the end of the incline.

**Homework assignment 3:** Chapter 1: 19

Chapter 2: 14, 18, 20

Problem D: Consider a ladder of length $L$ standing against a wall making angle $\theta_0$ with the floor. At time $t = 0$ the walls and the floor become perfectly frictionless and the ladder starts to slide down. Using the angle $\theta$ between the ladder and the floor as the generalized coordinate find the equation of motion (the differential equation) of the ladder and calculate, in terms of $\theta_0$, the angle at which the ladder leaves the wall (remember the sliding rod experiment we did in class.) You will need to review the concept of moment of inertia and find the moment of inertia of the ladder about its center of mass.

Problem E: Consider a mass $m$ sliding without friction on a spiral given by

$$x = a\cos\theta \quad ; \quad y = a\sin\theta \quad ; \quad z = \epsilon a\theta$$

Let gravity point in the -$z$ direction. Using the angle $\theta$ as the generalized coordinate solve for $\theta(t)$ if the initial conditions are $\theta(0) = 0$ and $\dot{\theta}(0) = 0$. Then find the radial and vertical constraint forces $F_r(t)$ and $F_z(t)$ that keep the particle on the spiral. Use the method of Lagrange multipliers in conjunction with Newton's second law to connect the Lagrange multipliers with the constraint forces.

Hints and clarifications:

2.14: To find the normal force of constraint you will need to let the radial position of the center of mass be a variable (say $\rho$) with an associated Lagrange multiplier.

2.18: Goldstein's $h$ is a constant of the motion, but not necessarily the energy. Show in this case that $h$ is not the energy.

2.20: I suggest that you use $x$ and $y$ for $m$, then $X$ for $M$, nice square Cartesian coordinates. You should find that the forces of constraint do no work.

Problem D: The ladder leaves the wall when $\sin\theta = (2/3)\sin\theta_0$.

**Homework assignment 4:** Chapter 4: 1, 3, 6, 7, 14, 15, 20, 21, 24, 25.

Problem J: Consider a rotation for which the Euler angles are $(\phi = 1, \theta = 1, \psi = 1)$. Euler's theorem says that this compound rotation is a single rotation about some axis by some angle $\Omega$. Find the direction in $(x, y, z)$ space of this axis and give the value of the rotation angle $\Omega$.

**Homework assignment 5:** Chapter 5: 3, 11, 14, 15, 17, 18, 29

Problem K: Consider a chalkboard eraser with length $L$, width $w$, and thickness $h$ (all three lengths are different.)

(a) Find an eraser, or a similar object with three different rectangular dimensions, and try spinning it about each of its principal axes. Verify experimentally that about two of the axes stable spinning motion is observed, but that about one of them the spinning is unstable in the sense that the object begins to tumble, meaning that components of ($\boldsymbol{\omega}$ other than the one you started with become large.

(b) Use the Euler equations to explain the behavior you observed in (a) and tell which of the three principal axes is the unstable one.

Hints and clarifications:

In all problems, remember and believe that $\boldsymbol{\omega}$ is a vector.

In Problem 17 remember what we did in class, find $\boldsymbol{\omega}$ again, and then when you find $\mathbf{L}$ and $T$, use $\alpha = \pi/4$ and express the vector components of $\mathbf{L}$ in a coordinate system in which $\mathbf{z}$ points straight up from the table and $\mathbf{x}$ points horizontally and opposite to the long axis of the cone.

In Problem K: Look at the motion about each of the principal axes separately by assuming (in the case of motion mostly about the $x$-axis) $\omega_2 \approx \omega_3 \ll \omega_1$, then linearizing Euler's equations to see if the equations of motion for $\omega_2$ and $\omega_3$ give stability or instability. Repeat for the other two cases.

**Homework assignment 6:** Chapter 5: 10
Chapter 6: 6.4, 6.6(b)

Problem M: Let the right-circular cone of Problem 5.17 rotate without slipping on a flat table tilted by an angle $\beta$ above the horizontal (a horizontal table has $\beta = 0$). (Use a half-angle of $\pi/4$ again.) Choose a suitable general coordinate and find the equation of motion of the cone in terms of this coordinate using the Lagrangian. Find the frequency of small amplitude oscillations about the stable equilibrium point. Starting point: rather than rehashing the cone problem from last week you may start with the kinetic energy formula

$$T = \frac{7\pi}{80} \rho h^5 \Omega^2$$

where $\rho$ is the mass density of the cone, $h$ is the height of the cone from tip to base center, and where $\Omega$ is the rotation frequency of the center of mass of the cone about the axis perpendicular to the table.

Problem N: (a) Consider two point charges of charge $Q$ and mass $M$ with one fixed at the origin and the other free to move on the positive $z$ axis, with $z$ vertical. Find the equilibrium position of the free charge under the combined effects of gravity and electrostatic repulsion, and then find the frequency of small oscillations about equilibrium. Express your frequency in terms of $g$ and the equilibrium position $z_0$.

(b) Repeat part (a), but with two free charges on the positive $z$ axis. Find the two modes of low amplitude vibration and their frequencies. (In this part you will need to find the equilibrium numerically, so let the point charges be alpha particles with $Q = 1.6 \times 10^{-19}$C and $M = 6.6 \times 10^{-27}$kg.)

Problem P: Consider 4 equal masses $m$ constrained to move on a circle. Each mass is attached to its neighbors by linear springs of spring constant $k$ (the springs don't follow the circle– they are along the lines joining the masses). Let the springs have a relaxed length of $R$, where $R$ is the radius of the circle. Find the normal modes of vibration of these 4 masses and their corresponding frequencies about the equilibrium configuration of the 4 masses.

Problem Q (Maple writing assignment): Consider three pendulums side by side, each with length $L$ and mass $m$. They are connected by springs each with spring constant $k$ such that when the pendulums hang straight down the springs are relaxed. Find the normal modes and frequencies for this system.

Hints and clarifications:

Problem 6.4: Use the full matrix machinery and the normal coordinates with initial conditions $\theta_1(0) = 0.1$, $\theta_2 = 0$ and both angular velocities set to zero. You will probably need to use Maple and should use the following equations: 6.4, 6.6, 6.23, 6.35, and 6.39. To make things simple, show that all of the frequencies come out in units of $\sqrt{g/L}$, which means that you get to set $m = 1$, $L = 1$, and $g = 1$ when you do the linear algebra details because at the end you know that the numerical frequencies you will obtain are multiplied by $\sqrt{g/L}$.

Problem 6.6: Don't worry about the hint in (a); just use Maple (his part (b)) and do the problem. You will get a mess, so after you get it set $m = 1$, $M = 3$, and $k = 1$, then look at each frequency and eigenvector and make a qualitative explanation about why some modes have higher frequencies than others.

**Homework assignment 7:** Chapter 7: 7.1, 7.3, 7.7, 7.13, 7.17, 7.19, 7.21, 7.24

**Homework assignment 8:** Chapter 8: 8.2, 8.14, 8.15, 8.16, 8.19, 8.26

Problem R: Using the relativistic Hamiltonian, find Hamilton's equations of motion in the variables $(r, \theta)$ for motion in the plane under the influence of a central force potential $V(r)$. Assume that the heavy attracting body has infinite mass so that the reduced mass is the mass of the smaller body.

**Homework assignment 9:** Chapter 9: 9.2, 9.4, 9.6, 9.10, 9.24, 9.34, 9.41

9.34: Using Eq. 9.116 really means reading through all of the infinitesimal contact transformation discussion and then using Eq. 9.119 to finally do the harmonic oscillator calculation. Again, use

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2$$

and let the initial conditions be $q(0) = 1$ and $p(0) = 1$.

Problem 9.41: There is nothing to do in this problem (I don't think) unless you change the modified Hamiltonian to this:

$$H = H_0(q,p) - \epsilon q \sin \omega t$$

Restoring the equations to canonical form means to find canonical variables $(Q, P)$ such that the new Hamiltonian is

$$K = H_0(q(Q,P,t), p(Q,P,t))$$

i.e., the explicit time-dependent term is gone from the the new Hamiltonian (the sine-function is buried inside $H_0$ now.)

In part (c) let $H_0 = p^2/2$ (free particle) and discuss the meaning of the new variables.

Part (d): Extra Credit. Find a canonical transformation that converts the equations of motion to

$$\dot{Q} = P \quad ; \quad \dot{P} = 0$$

when the original Hamiltonian is the free particle Hamiltonian

$$H_0 = \frac{1}{2}p^2$$

Note: this *does not* mean that $K = P^2/2$. The new Hamiltonian could have an additive time-dependent term and still produce the simple equations of motion given above.

**Homework assignment 10:** Chapter 10: 10.5, 10.7, 10.8, 10.15, 10.27

**Problem S (Maple writing assignment)**: Consider the Hamiltonian for 2-d projectile motion, with gravity in the $-y$-direction and no force in the $x$-direction (in standard dimensionless form with $m = 1$ and $g = 1$):

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + y$$

with initial conditions $y(0) = 1$, $p_y(0) = 0$, $x(0) = 0$, and $p_x = 2$.

You will do this problem in two different ways, corresponding to two different ways of implementing the use of constants of integration as described in the second paragraph on page 442.

(a) Write down the restricted Hamiltonian-Jacobi equation, Eq. (10.43) for this problem and carry through the calculations indicated in the right-hand column on pages 442 and 443. Choose

$$W(x,y) = W_x(x,\gamma_2) + W_y(y,\gamma_1) \quad \text{with} \quad P_y = \gamma_1 \;\; , \;\; P_x = \gamma_2$$

where $(\gamma_1, \gamma_2)$ are the two separation constants

$$\frac{1}{2}\left(\frac{\partial W_x}{\partial x}\right)^2 = \gamma_2 \quad ; \quad \frac{1}{2}\left(\frac{\partial W_y}{\partial y}\right)^2 + y = \gamma_1$$

so that $\alpha_1 = \gamma_1 + \gamma_2 \Longrightarrow K = P_1 + P_2$. Use the Hamilton-Jacobi theory to find $q(t)$ and $p(t)$.

(b) Now do the problem again, but this time choose

$$W(x, y) = W_x(x, \gamma_2) + W_y(y, \alpha_1, \gamma_2) \quad \text{with} \quad P_y = \alpha_1 \ , \ P_x = \gamma_2$$

where $(\alpha_1, \gamma_2)$ are the two separation constants

$$\frac{1}{2}\left(\frac{\partial W_x}{\partial x}\right)^2 = \gamma_2 \quad ; \quad \frac{1}{2}\left(\frac{\partial W_y}{\partial y}\right)^2 + y = \alpha_1 - \gamma_2$$

so that $K = P_y$, as described in Eq. (10.46). Use the Hamilton-Jacobi theory again to find $q(t)$ and $p(t)$, and along the way show that Hamilton's equation of motion for $Q_x$ does not involve the time at all (as described at the top of page 442) and hence gives an orbit equation $y(x)$.

**Problem T (Maple writing assignment)**: Consider the family of power-law potentials of the form

$$V(x) = V_0 \left|\frac{x}{a}\right|^p$$

Assume that $p > 0$ so that the potential is a well in which a particle can oscillate. Show, using the action-angle formalism, that for such a potential the oscillation frequency $\nu$ depends on the energy according to a power law

$$\nu = BH^r$$

and find the relation between $p$ and $r$ as well as the constant $B$. In particular, show that for $p = 2$ and $p = 1$ (we did this one in class) that you get the expected answer.

Problem U: For each of the following potentials, find either $H(J)$ or $J(H)$ and use your result to find $\nu(H)$, the variation of the frequency with oscillator energy.

(a)

$$V(x) = -\frac{k}{|x|}$$

for a particle of mass $m$.

(b) A particle confined to a one dimensional box of length $L$, bouncing back and forth. (After you get this simple result, make sure that it agrees with the correct limiting case in Problem S.) Let the particle have mass $m$.

(c) The pendulum,

$$V(\theta) = 2MgL\sin^2\left(\frac{\theta}{2}\right)$$

Find $J(H)$ for both oscillation and rotation assuming that the moment of inertia is $I = ML^2$. Also find $nu(H)$ for both of these cases. Show that your complicated elliptic integral answers reduce to the correct forms for small oscillations and for high energy rotation.

Hints and clarifications: Problem 10.15: Answer:

$$J = 2\pi x_0 \sqrt{2ma} \left( \sqrt{\frac{H}{a}} - 1 \right)$$

Problem S: In each case you should find $x = 2t$, $p_x = 2$, $y = 1 - t^2/2$, $p_y = -t$, $y = 1 - x^2/8$. Note: As we have done Hamilton-Jacobi theory we have always assume that when you have

$$\left( \frac{\partial W}{\partial q} \right)^2 = F \quad \text{then} \quad \frac{\partial W}{\partial q} = \sqrt{F}$$

But $\partial W/\partial q = p$, and if $p$ is negative, you will get the sign wrong by always taking the positive square root. So watch for this problem to crop up in Problem R, and when it does, realize that it is OK to use $-\sqrt{F}$ when appropriate.

Problem 10.5: Since he doesn't give you initial conditions you are finished when you have something like this for $q$ and $p$:

$$A \cos \omega t + B \sin \omega t$$

Problem 10.7: You will need to go all the way back to the Lagrangian to find the generalized momenta $p_v$ and $p_u$ so that you can build the Hamiltonian. You should find the factors $\sinh v^2 + \sin u^2$ and $\sinh v^2 \sin u^2$ showing up. To decide what form the potential can have, use the Staeckel conditions.

To reduce the problem to quadratures in the last part all I want you to do is to put the Hamilton-Jacobi equation for $W$ into the form

$$[...]_v + [...]_u = 0$$

where $[...]_v$ only contains dependence on $v$ and $[...]_u$ only contains dependence on $u$. Once you get to here, show that it is legal to set $W = W_v + W_u + W_\phi$ to obtain $[...] = \alpha$ with a separate $\alpha$ for each [...] so that you can solve for $\partial W/\partial u$ and $\partial W/\partial v$ and integrate to obtain each $W$. This is what is meant by reducing the problem to quadratures.

Problem 10.8: This one doesn't separate because of the $xt$ term. But $xt$ is so simple that it is possible to find $S(x, \alpha, t)$ by playing around with terms like $x$, $t$, $t^3$, $xt^2$, etc. I used Maple to try forms for $S$ where these terms were added together with arbitrary constants multiplying them, then substituted my guess into the Hamilton-Jacobi equation to see if a solution could be found. It is possible, but it is a pretty good puzzle to find the answer. Notice that when you get done you need to have one arbitrary constant ($\alpha$) left undetermined so that it can be the new momentum $P$.

In Problem 27: "Express the motion in terms of $J$ and its conjugate angle variable." means to make your answer look like the harmonic oscillator expressions in Eqs. 10.96-97. As you do this problem, first find the conditions for a circular orbit and write down the circular orbit Hamiltonian in action angle form using $J_\theta$ for the circular orbit. Then allow $r$ to deviate slightly from the circular orbit, $r = r_0 + r_1$ and expand in small $r_1$ to obtain a perturbed Hamiltonian involving $p_{r1}$, $r_1$, and $p_\theta$. Assume that $p_\theta$ is unchanged and that $J_\theta$ is the same as in the circular orbit case and solve for $J_r$, finally expressing the perturbed Hamiltonian in terms of $J_r$ and $J_\theta$. Then use this perturbed Hamiltonian to find the frequency of the radial oscillations about the circular orbit.

Problem T: Change the integration variable from $x$ to something dimensionless so that the integration limits are from 0 to 1 and all of the physical variables are in a factor multiplying the integral. This makes it possible to see how the frequency varies with energy without worrying about some horrible function you have never heard of.

Problem U: (a) Use the same dimensionless trick as in Problem S. (b) This is easy. (c) Maple will do this problem, giving you answers in terms of elliptic integrals $E$ and $K$. You will need to help Maple out by using the `assume` command. You may need to tell it the range of $H$ you are using and that $M$, $g$, and $L$ are positive. Partial answer: for oscillation

$$J = 16ML\sqrt{gL}\left[E\left(\sqrt{\frac{H}{2MgL}}\right) - K\left(\sqrt{\frac{H}{2MgL}}\right)\left(1 - \frac{H}{2MgL}\right)\right]$$

**Homework assignment 11:** Chapter 11: 11.4, 11.16 Chapter 12: 12.4, 12.9, 12.10

Problem V: Consider the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{\epsilon}{3}q^3$$

(a) With initial conditions $q(0) = 0$, $p(0) = 1$, use an iterative scheme followed by a Taylor expansion to find the two turning points of this motion through second order in $\epsilon$ using conservation of energy in the Hamiltonian directly (no dynamics). Partial answer: the right turning point is at

$$q = 1 - \frac{1}{3}\epsilon + \frac{5}{18}\epsilon^2$$

(b) Use the Hamilton-Jacobi perturbation method given on the Maple handout I put in your box to find the order $\epsilon$ correction to the simple-harmonic oscillator approximation to the motion in this well. Verify that you get the turning points right and if there is a frequency shift, find it.

(c) Compare your approximate motion to the exact motion by making the same kind of plots that I made on the handout.

(d) Read through the example starting on page 545 of Goldstein and notice that his example is this problem, with the proper choice of the constants. Use the final results of this example on pages 546 and 547 to calculate the second-order frequency shift for our Problem V and verify that it is correct by carefully using your numerical solution in part (c).

In Problem 16 I would also like you to solve the equation for the two values of the 2-cycle, then evaluate the derivative of $F(F(x))$ at these values to determine the critical value of $a$ between 3 and 4 at which the 2-cycle becomes unstable.

**Homework assignment 12:** 13.1

Problem W: If we let $\delta p$ be the perturbed pressure in an acoustic wave an appropriate Lagrangian density is

$$\mathcal{L} = \frac{\rho_0}{2}\left(\frac{\partial \delta p}{\partial t}\right)^2 - \frac{\gamma p_0}{2}(\nabla \delta p)^2$$

Using the expression for the gradient in spherical coordinates, and remembering that the volume element in spherical coordinates is

$$dV = r^2 \sin\theta d\phi d\theta dr$$

derive the acoustic wave equation in spherical coordinates by finding the Euler-Lagrange equation for this density.