

## Important Stuff From Physics 721

Text: Goldstein, Poole, and Safko

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### Chapter 1: Elementary Principles

#### The axioms of classical mechanics:

I: Inertial frames exist, and in these frames a single particle moves with constant velocity. This constant-velocity motion defines the uniform flow of time.

II: When two particles interact the total momentum is the same at all times:  $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{constant}$ .

III: If a particle accelerates we say that a force has caused it, with the force defined to be:

$$\mathbf{F} = m\mathbf{a}$$

Note that these three are not mathematical results. I and II are based on experiments and III is a definition. In an ugly complicated world axiom III, the famous Newton's Second Law, would be worthless because  $\mathbf{F}$  might depend strongly on time of day, day of the month, leap year, phases of Venus, sunspot cycle, etc. But there is apparently order in our world and in nearly all cases we have  $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v})$  with a rather simple functional dependence on position and velocity, at least approximately.

The basic problem of dynamics, then, is to solve problems of the form

$$m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r}, \mathbf{v})$$

We will do so this semester with ever-increasing levels of sophistication.

#### Dynamical variables:

##### Momentum $\mathbf{p}$ :

$$\mathbf{p} = m\dot{\mathbf{r}} \quad , \quad \frac{d\mathbf{p}}{dt} = \mathbf{F} \quad (\text{good for variable mass problems})$$

It is conserved in the absence of a force.

##### Angular momentum $\mathbf{L}$ and torque $\mathbf{N}$ :

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad , \quad \frac{d\mathbf{L}}{dt} = \mathbf{N} = \mathbf{r} \times \mathbf{F}$$

It is conserved in the absence of torque.

##### Energy:

$$\text{if } \mathbf{F} = -\nabla V \text{ then } m\ddot{\mathbf{r}} = -\nabla V \Rightarrow m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\dot{\mathbf{r}} \cdot \nabla V \Rightarrow \frac{d}{dt} \left( \frac{1}{2}mv^2 + V \right) = \frac{\partial V}{\partial t} \quad , \quad E = \frac{1}{2}mv^2 + V$$

It is conserved when the potential is not an explicit function of time, i.e., when the potential only changes in time because  $\mathbf{r}$  changes in time.

##### Work and power:

$$W = \int \mathbf{F} \cdot d\mathbf{r} \quad ; \quad P = \mathbf{F} \cdot \mathbf{v}$$

Any force that always acts perpendicular to the velocity of a particle cannot change its energy.

##### Many particles:

The net force on a collection of many particles resulting from inter-particle interactions is always zero, i.e., you can't pick yourself up by pulling on your belt. If the total external force acting on the system is  $\mathbf{F}^{(e)} = \sum \mathbf{F}_i^{(e)}$  and if the total

mass is  $M = \sum m_i$  then we have

$$\mathbf{R} = \frac{1}{M} \sum m_i \mathbf{r}_i \quad \text{is the position of the center of mass, and} \quad \mathbf{F}^{(e)} = M\ddot{\mathbf{R}}$$

It is usually convenient to decompose the position vector  $\mathbf{r}_i$  of each particle into the position  $\mathbf{R}$  of the center of mass plus the position vector  $\mathbf{r}'_i$  from the center of mass to the particle position:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i$$

**Momentum:**

$$\mathbf{P} = \sum m_i \mathbf{v}_i = M\dot{\mathbf{R}} = M\mathbf{V}$$

i.e., the total momentum of the collection is just the total mass multiplied by the velocity of the center of mass.

**Angular momentum:**

$$\mathbf{L} = M\mathbf{R} \times \mathbf{V} + \sum m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i = M\mathbf{R} \times \mathbf{V} + I\boldsymbol{\omega}$$

where  $I$  is the moment of inertia about the center of mass and where  $\boldsymbol{\omega}$  is the angular rotation vector (to be discussed in Chapter 4).

**Kinetic energy:**

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum m_i v_i'^2 = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$$

**Constrained motion:**

A constraint condition takes the form

$$f(\mathbf{x}, t) = 0$$

and reduces the dimensionality of the space in which a particle moves by one, e.g., one constraint in  $(x_1, x_2, x_3)$  makes the particle move on a surface and two make it move on a space curve.

**Normal to the constraint surface:**

$$\mathbf{n} \propto \nabla f$$

**Constraint force:**

If the force of constraint  $\mathbf{C}$  does no work on the moving particle then

$$\mathbf{C} = \lambda(t)\nabla f$$

**Generalized coordinates:**

The easiest way to implement a constraint is to trade  $(x_1, x_2, x_3)$  in on new coordinates that exactly fit the constrained motion, e.g., use  $\theta$  for a plane pendulum,  $(\theta, \phi)$  for a spherical pendulum, etc.

**The Lagrangian:**

$$T(q, \dot{q}) = \frac{m}{2}v^2 = \frac{m}{2} \frac{d\boldsymbol{\ell}}{dt} \cdot \frac{d\boldsymbol{\ell}}{dt} \quad ; \quad L = T(q, \dot{q}) - V(q, t)$$

Know how to handle cylindrical coordinates, spherical coordinates, and motion along space curves parametrically determined in the form  $(x_1(s), x_2(s), x_3(s))$ . For example,  $x = as$  and  $y = as^2$  gives

$$T = \frac{m}{2}a^2\dot{s}^2(1 + 4s^2)$$

**Euler-Lagrange equations of motion:**

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

**Equivalent Lagrangians:**

If two Lagrangians differ by a total time derivative, i.e., if there exists some function of position  $\Phi(q_k, t)$  such that

$$L_2 = L_1 + \frac{d}{dt}\Phi(q_k, t)$$

then the Euler-Lagrange equations for both  $L_1$  and  $L_2$  give the same equations of motion.

**Conservation law:** If  $L$  is not an explicit function of time, i.e, if  $L = L(q_k, \dot{q}_k)$ , then the quantity

$$h = \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

is conserved. This quantity is often, but not always, the energy  $E = T + V$ .

**Electromagnetic Lagrangian:** To describe particle motion in electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  given by

$$\mathbf{E} = -\nabla\varphi(q_k, t) - \frac{\partial\mathbf{A}(q_k, t)}{\partial t} \quad ; \quad \mathbf{B} = \nabla \times \mathbf{A}(q_k, t)$$

use the Lagrangian

$$L = T - e\varphi + e\mathbf{v} \cdot \mathbf{A}$$

**Chapter 2: Calculus of Variations****Lagrange multipliers:**

To find the extremum (max or min) of a function of a few variables, subject to a constraint, it is often easier to add one more variable  $\lambda$  (called a Lagrange multiplier) to the problem. Suppose, for example, that we want to minimize  $F(a, b)$  subject to the constraint that  $G(a, b) = \Gamma$ . The method of Lagrange multipliers has us minimize the new function

$$H(a, b, \lambda) = F(a, b) + \lambda(G(a, b) - \Gamma)$$

by solving the three equations

$$\frac{\partial H}{\partial a} = 0 \quad ; \quad \frac{\partial H}{\partial b} = 0 \quad ; \quad \frac{\partial H}{\partial \lambda} = 0$$

**Functional extremum problems (calculus of variations):**

Suppose that we want to consider all possible functions  $y(x)$  that all have the same endpoint values at  $x = x_1$  and  $x = x_2$ . Among this infinity of possibilities we now want to find the function  $y(x)$  that either maximizes or minimizes the value of the integral

$$I = \int_{x_1}^{x_2} F(y, y', x) dx$$

Unless  $F(y, y', x)$  is of the right form this problem may not have a well-defined solution, but for a well-posed problem the desired function is given as the solution of the Euler-Lagrange differential equation

$$\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0$$

**First integral (like energy conservation):**

If  $F$  does not depend explicitly on  $x$ , i.e., if it is of the form  $F(y, y')$  then the second-order Euler-Lagrange differential equation has a first integral, which is easier to solve:

$$y' \frac{\partial F}{\partial y'} - F = c, \quad \text{a constant}$$

Always take advantage of this simplified equation whenever it is available.

### Adding a constraint:

If we want  $y(x)$  that maximizes or minimizes an integral, subject to an integral constraint, we use the method of Lagrange multipliers. For instance, if we want to find the extremum of  $I$  subject to the integral constraint

$$K = \int_{x_1}^{x_2} G(y, y', x) dx - \Gamma = 0$$

then we invent the new functional  $I + \lambda K$  given by

$$I + \lambda K = \int_{x_1}^{x_2} [F(y, y', x) + \lambda G(y, y', x)] dx - \lambda \Gamma$$

in which the new integrand is now

$$H(y, y', x) = F(y, y', x) + \lambda G(y, y', x)$$

We now find the solution of the problem by solving the equations

$$\frac{d}{dx} \left( \frac{\partial H}{\partial y'} \right) - \frac{\partial H}{\partial y} = 0 \quad ; \quad \frac{\partial (I + \lambda K)}{\partial \lambda} = 0 \Rightarrow K = 0$$

again remembering to use the first integral if it is available.

**Hamilton's principle:** Newton's second law is equivalent to the statement that a particle moves along the path  $q(t)$  in such a way that the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

is minimized. The calculus of variations immediately leads to the standard Euler-Lagrange equation of motion:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

### The connection to quantum mechanics:

In Feynman's path-integral formulation of quantum mechanics the basic principle is that the probability amplitude  $\psi(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)$  for a particle that is found at time  $t_1$  at position  $\mathbf{x}_1$  to later be found at time  $t_2$  at position  $\mathbf{x}_2$  is proportional to the horrible infinite sum of the complex functional

$$\exp \left( \frac{i}{\hbar} \int_{t_1}^{t_2} L dt \right)$$

taken over all possible paths from  $\mathbf{x}_1$  to  $\mathbf{x}_2$ . For macroscopic objects the magnitude of  $L/\hbar$  is astronomical, so the sum will phase average to zero everywhere except along a very narrow path where nearby paths all have the same value of the action, i.e., along the path where the action is a minimum or maximum. This is the nicest way I know to qualitatively see how to get from quantum mechanics to  $\mathbf{F} = m\mathbf{a}$ : Feynman's quantum mechanics  $\Rightarrow$  Hamilton's principle  $\Rightarrow$  Euler-Lagrange equations of motion  $\Rightarrow$  Newton's second law (See *Newtonian Mechanics*, McGraw-Hill Inc., 1983, by Ralph Baierlein, Chapter 4.)

### Lagrangian dynamics with constraints:

Usually when we use the Lagrangian approach we choose coordinates that fit with the constraints, but this isn't necessary. Alternatively, the constraints could be included by using the method of Lagrange multipliers, with the slight complication that the multiplier(s)  $\lambda$  are now functions of time to be determined as part of the solution to the problem. For  $K$  time-independent constraints of the form  $f_I(q_k) = 0$  the constraint-modified Lagrangian is

$$\mathcal{L} = L + \sum_{I=1}^K \lambda_I(t) f_I(q_k)$$

If there are  $K$  constraints and  $n$  coordinates there are now  $n+K$  unknown functions to find (the  $q_k(t)$  and the  $\lambda_I(t)$ ). The Euler-Lagrange equations give  $n$  equations and the equations of constraint provide the additional  $K$  equations needed to determine the solution.

For time-independent constraints that do not involve the  $\dot{q}_k$  the equations of motion are of the form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda \frac{\partial f}{\partial q}$$

When a dynamics problem is solved in this way the Lagrange multiplier functions  $\lambda(t)$  are proportional to the forces of constraint. To see exactly how, put your equations of motion in the form

$$m\mathbf{a} = \dots$$

so that the right hand-side terms are forces. The terms containing the Lagrange multipliers are the constraint forces.

### Noether's theorem:

When the Lagrangian has a "symmetry", i.e., when some change of coordinates leaves the form of the Lagrangian the same, there is always a constant of the motion associated with this symmetry. We have already seen two examples of this. (1) In Cartesian coordinates if the Lagrangian doesn't depend on  $x$  then the Euler-Lagrange equation for  $x$  is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad m\dot{x} = \text{constant} \quad ,$$

i.e., linear momentum is conserved. (2) In plane polar coordinates when the Lagrangian doesn't depend on  $\theta$  then

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad mr^2\dot{\theta} = \text{constant} \quad ,$$

i.e., the  $z$ -component of the angular momentum is conserved.

**Rayleigh damping:** An easy way to add simple linear damping to equations of motion obtained from the Lagrangian formalism is to define the Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2} \sum b_i v_i^2$$

expressed in terms of the generalized coordinates  $q_k$  and then derive the equations of motion as usual from the Lagrangian, but with an extra term added:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{\partial \mathcal{F}}{\partial \dot{q}_k} = 0$$

## Chapter 3: Central Force Problems

**Central forces:** Consider two masses  $m_1$  and  $m_2$  located at positions  $\mathbf{r}_1$  and  $\mathbf{r}_2$  interacting via a force which depends only on the magnitude of the distance between them and which points along the line connecting them.

The center of mass

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

moves at constant velocity while the difference vector

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

traces out a 2-dimensional orbit in the plane perpendicular to the conserved total angular momentum vector. Note that

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2} \mathbf{r} \quad ; \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2} \mathbf{r}$$

so that if  $\mathbf{r}$  traces out some shape in space,  $\mathbf{r}_1$  and  $\mathbf{r}_2$  trace out the same shape, just with different sizes and always sitting opposite each other across the position of the center of mass.

Using cylindrical coordinates in this plane leads to the conserved relative-motion angular momentum

$$\ell = \mu r^2 \dot{\theta}$$

with the reduced mass  $\mu$  given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

and to an effective radial potential given by

$$V_{\text{eff}} = \frac{\ell^2}{2\mu r^2} + V(r) \quad .$$

The Lagrangian  $L$  and conserved energy  $E$  are given by

$$L = \frac{\mu}{2} \dot{r}^2 - V_{\text{eff}}(r) \quad ; \quad E = \frac{\mu}{2} \dot{r}^2 + V_{\text{eff}}(r)$$

Orbit differential equation:

$$u = \frac{1}{r(\theta)} \quad ; \quad \frac{d^2 u}{d\theta^2} + u = -\frac{m}{\ell^2} \frac{d}{du} V\left(\frac{1}{u}\right)$$

### Kepler's laws:

(1) All central force-law potentials obey the rule that the difference vector  $\mathbf{r}$  sweeps out equal areas in equal times. This is simply a consequence of angular momentum conservation since

$$\frac{dA}{dt} = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu}$$

(2) For the attractive inverse-square law potential  $V(r) = -k/r$  the vector  $\mathbf{r}$  traces out a conic section with the origin  $r = 0$  at one focus:

$$r(\theta) = \frac{\ell^2/k\mu}{1 + \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \cos(\theta - \theta')} = \frac{a(1 - e^2)}{1 + e \cos(\theta - \theta')}$$

$$e = 0 \quad \Rightarrow \quad \text{circle}$$

$$0 < e < 1 \quad \Rightarrow \quad \text{ellipse}$$

$$e = 1 \quad \Rightarrow \quad \text{parabola}$$

$$e > 1 \quad \Rightarrow \quad \text{hyperbola}$$

Ellipse properties:

$$\text{Turning points : } r_1 = a(1 - e) \quad r_2 = a(1 + e) \quad \text{semimajoraxis : } a = (r_1 + r_2)/2 \quad \text{semiminor axis : } b = a\sqrt{1 - e^2}$$

$$\text{distance from ellipse center to focus : } f = ae \quad ; \quad \text{Cartesian form (relative to the ellipse center) : } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Useful relation:

$$E = -\frac{k}{2a}$$

The above formula for  $r(\theta)$  coupled with conservation of angular momentum  $\ell = \mu r^2 \dot{\theta} = \mu r v_\theta$  are the key to everything you might want to know about dynamics on the orbit. For example, if you needed to know  $\dot{r}$  you would get it this way:

$$\dot{r} = \frac{dr}{d\theta} \dot{\theta} = \frac{ke}{\ell} \sin \theta$$

(3) For the inverse-square law the square of the period is proportional to the cube of the semi-major axis  $a$ , or more precisely

$$T = 2\pi \sqrt{\frac{\mu}{k}} a^{3/2}$$

Parametric form for  $r(t)$  and  $\theta(t)$ :

Given time  $t$  solve Kepler's equation for  $\psi$ :

$$\psi - e \sin \psi = \omega t \quad \text{with} \quad \omega = \sqrt{\frac{k}{\mu a^3}}$$

Then find  $r(t)$  and  $\theta(t)$  from

$$r = a(1 - e \cos \psi(t)) \quad \cos \theta = \frac{\cos \psi(t) - e}{1 - e \cos \psi(t)}$$

## Chapter 6: Oscillations

### Small oscillations: standard procedure:

If you are given a 1-dimensional potential  $V(q)$ , here is the standard procedure I expect you to know how to follow to analyze the various possible motions of the system.

- (i) Write down the Lagrangian and find the Euler-Lagrange equations of motion.
- (ii) Set all of the time derivatives in the equation to zero and solve for the points of equilibrium:

$$\frac{\partial V}{\partial q} = 0$$

(iii) Perform a Taylor series about each equilibrium point in the equation of motion to discover whether the point is stable or unstable, then find either the linear growth  $\gamma$  or the linear frequency  $\omega$  for motion near each equilibrium point.

(v) Use conservation of energy to find the nonlinear period for larger amplitude motion about each equilibrium point. We practiced doing this in Chapter 1.

### Coupled oscillations:

When the motion of multiple particles is coupled we find the system equilibrium by solving for coordinates  $q_{0i}$  from the set of equations

$$\frac{\partial V}{\partial q_i} = 0$$

(these equations are often hard to solve). We then proceed as follows:

- (i) Look for small vibrations about the equilibrium by writing

$$q_i = q_{0i} + \eta_i$$

- (ii) Write down the kinetic energy in the form

$$T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j$$

where

$$T_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}$$

evaluated at equilibrium. Note that this forces  $T_{ij}$  to be symmetric, so if, for example,

$$T = 2x_1^2 + x_2^2 + 2x_1x_2 \quad ,$$

then the diagonal elements could be read right off:  $T_{11} = 2$  and  $T_{22} = 1$ , but the cross term is split:  $T_{12} = 1$  and  $T_{21} = 1$ .

Normally, however, the matrix  $\mathbf{T}$  is diagonal, so that in Cartesian coordinates it looks like this:

$$\begin{bmatrix} m_1 & 0 & 0 & \dots & 0 \\ 0 & m_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & m_n \end{bmatrix}$$

- (iii) Write down the potential energy in the form

$$V = \frac{1}{2} V_{ij} \eta_i \eta_j$$

where

$$V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

evaluated at the equilibrium point.

- (iv) The Euler-Lagrange equations for this system then take the form

$$T_{ij} \ddot{\eta}_j = -V_{ij} \eta_j$$

and if we now seek solutions of the form

$$\boldsymbol{\eta} = \mathbf{a}e^{-i\omega t}$$

then the mode frequencies  $\omega$  and the mode eigenvectors  $\mathbf{a}$  are obtained as the solutions of the generalized eigenvalue problem

$$\mathbf{V} \cdot \mathbf{a} = \lambda \mathbf{T} \mathbf{a} \quad \text{with} \quad \lambda = \omega^2$$

The best way to normalize the normal mode eigenfunctions is according to the rule

$$\mathbf{a}_j \cdot \mathbf{T} \cdot \mathbf{a}_j = 1$$

or, using the matrix  $\mathbf{A}$ , whose columns are the eigenvectors  $\mathbf{a}$ ,

$$\mathbf{A}^T \mathbf{T} \mathbf{A} = \mathbf{I}$$

The general solution can then be written

$$\boldsymbol{\eta} = \sum_k C_k \mathbf{a}_k e^{-i\omega_k t}$$

where the complex constants  $C_k$  are to be determined by the initial conditions.

## Chapter 7: Special Relativity

### Basic Postulates:

1. The laws of physics are the same to all inertial observers.
2. The speed of light is the same to all inertial observers.

### Spacetime Interval:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

is the same in all inertial frames, where the differentials  $dt, dx, dy, dz$  denote separations in time and space between two events.

### Time Dilation and Length Contraction:

For clocks ticking off time intervals of length  $\Delta\tau$  (in the rest frame of the clock) and for sticks of length  $L$  (in the frame in which the stick is at rest together with the clock), the time interval and length observed by someone who sees them go by at constant speed  $v$  are given by

$$\Delta t' = \gamma \Delta\tau \quad \text{and} \quad L' = L/\gamma$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{with} \quad \beta = \frac{v}{c}$$

### Lorentz Transformation:

Consider parallel Cartesian reference frames  $S$  and  $S'$  with their  $x$ -axes aligned, with their origins coincident at  $t = t' = 0$ , and with  $S'$  moving at speed  $v$  along the  $x$ -axis relative to  $S$ . Consider also an event which occurs in frame  $S$  at  $(t, x, y, z)$ . This same event will be observed to occur in frame  $S'$  at  $(t', x', y', z')$  given by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

### Velocity Addition:

Parallel boosts: If object A is moving along the  $x$ -axis with relativistic factor  $\beta$  relative to frame  $S$ , and if object B is moving at  $\beta'$  relative to object A, then the relativistic factor  $\beta''$  of object B as observed from frame  $S$  is given by

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

Non-parallel boosts: If object A is moving along the  $x$ -axis relative to  $S$ , and if object B is moving along some other axis relative to the frame of A, then the transformation from  $S$  to the frame of B is not a simple boost. It is, rather, a combination of a boost and a rotation. This effect gives rise to the Thomas precession.

**4-Vectors:** There are physical quantities other than  $(ct, x, y, z)$  that transform between frames according to the Lorentz transformation. A few of them are:

Velocity:

$$u^\mu = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$$

with

$$\gamma = \frac{1}{\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)/c^2}}$$

Momentum-Energy:

$$p^\mu = (E/c, p_x, p_y, p_z) = (\gamma mc, \gamma mv_x, \gamma mv_y, \gamma mv_z)$$

The invariant Lorentz contraction of  $p^\mu$  gives the famous relation

$$E^2 = p^2 c^2 + m^2 c^4$$

Total energy and momentum are conserved in particle interactions.

**Newton's Second Law:**

In relativity dynamics are governed by the modified Newton's second law

$$\frac{d}{dt} \frac{m\mathbf{v}}{\sqrt{1 - \beta^2}} = \mathbf{F}$$

**Relativistic Lagrangian:**

$$L = -mc^2 \sqrt{1 - \beta^2} - V$$

valid in any generalized coordinates.

**Relativistic Hamiltonian:**

$$H = p_i \dot{q}_i - L$$

valid in any generalized coordinates, with

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

In Cartesian coordinates we have

$$H = \sqrt{p^2 c^2 + m^2 c^4} + V$$

## Chapter 8: Hamilton's Equations

**The Hamiltonian and Hamilton's equations:**

The Hamiltonian is defined by

$$H(q, p) = p_k \dot{q}_k - \hat{L}(q, p, t)$$

where  $\hat{L}(q, p, t) = L(q, \dot{q}(q, p, t), t)$  with the function  $\dot{q}(q, p, t)$  obtained by inverting the  $n$  equations that define the canonical angular momenta,  $p_k = \partial L / \partial \dot{q}_k$ . In the new canonical coordinates  $(q, p)$  the equations of motion are obtained as a first-order set known as Hamilton's equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad ; \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Note that if  $q_i$  is missing from the Lagrangian, it will also be missing from the Hamiltonian and the second of Hamilton's equations explicitly shows that the corresponding canonical momentum  $p_i$  is conserved.

**The  $\eta$  notation:**

The vector  $\eta$  for a system with  $n$  degrees of freedom is defined by

$$\eta = [\eta_1, \dots, \eta_n, \eta_{n+1}, \dots, \eta_{2n}] = [q_1, \dots, q_n, p_1, \dots, p_n]$$

Using this notation Hamilton's equations become

$$\dot{\eta}_j = J_{jk} \partial_k H$$

where the  $2n \times 2n$  symplectic matrix  $J_{jk} = J$  is defined to be

$$\begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0}_n \end{bmatrix}$$

where  $\mathbf{0}_n$  is the  $n \times n$  zero matrix and where  $\mathbf{I}_n$  is the  $n \times n$  identity matrix. This matrix has the following useful properties:

$$J^{-1} = -J \quad ; \quad J^T = -J \quad ; \quad J^2 = -\mathbf{I}$$

**Chapter 9: Canonical transformations**

**Canonical transformations:** A change of dynamical variables from  $(q, p)$  to  $(Q, P)$  defined by

$$Q = Q(q, p, t) \quad P = P(q, p, t)$$

is said to canonical if and only if there is some function  $K(Q, P, t)$  such that the same dynamics described by  $(q(t), p(t))$  generated by the Hamiltonian  $H(q, p, t)$  is given in the new variables by some new Hamiltonian  $K(Q, P, t)$ , i.e.,

$$\dot{Q}_k = \frac{\partial K}{\partial P_k} \quad ; \quad \dot{P}_k = -\frac{\partial K}{\partial Q_k}$$

**Generating functions:** A useful technique for finding canonical transformations is the concept of a generating function. The idea is that  $(q, p, H)$  and  $(Q, P, K)$  are related by

$$p\dot{q} - H = P\dot{Q} - K + \frac{dF}{dt}$$

where  $F$  is an arbitrary function  $F(q, p, Q, P, t)$ . There are four useful forms that this very general arbitrary function can take, and they are summarized in the table below, along with the rules for finding equations that connect  $(q, p)$  with  $(Q, P)$ . Note that in all cases the new Hamiltonian  $K(Q, P, t)$  is given by

$$K = H + \frac{\partial F_i}{\partial t}$$

$F = F_1(q, Q, t)$	$p_k = \frac{\partial F_1}{\partial q_k} \quad ; \quad P_k = -\frac{\partial F_1}{\partial Q_k}$
$F = F_2(q, P, t) - Q_k P_k$	$p_k = \frac{\partial F_2}{\partial q_k} \quad ; \quad Q_k = \frac{\partial F_2}{\partial P_k}$
$F = F_3(p, Q, t) + q_k p_k$	$q_k = -\frac{\partial F_3}{\partial p_k} \quad ; \quad P_k = -\frac{\partial F_3}{\partial Q_k}$
$F = F_4(p, P, t) + q_k p_k - Q_k P_k$	$q_k = -\frac{\partial F_4}{\partial p_k} \quad ; \quad Q_k = \frac{\partial F_4}{\partial P_k}$

If you are given half of a transformation (either  $Q(q, p, t)$  or  $P(q, p, t)$ ) try to partially invert it so that you have either  $q = \dots$ ,  $p = \dots$ ,  $Q = \dots$ , or  $P = \dots$  where  $\dots$  is a mixed function corresponding the one of the four types  $((q, Q)$ ,  $(q, P)$ , etc.) in the table. Then try to integrate to find the appropriate  $F_i$ . Then differentiate to get the other half of the transformation pair in the table and invert the equations to find  $q(Q, P, t)$ ,  $p(Q, P, t)$ ,  $Q(q, p, t)$ , and  $P(q, p, t)$ .

If you are clever you can choose the transformation in such a way that in  $(Q, P)$  the Hamiltonian is really simple, perhaps  $K = P$  or maybe even  $K = 0$ . Solve the new Hamiltonian equations for  $(Q(t), P(t))$  then use your transformation equations to find  $(q(t), p(t))$ .

**The symplectic approach:**

Using the  $\eta$  description of  $(q, p)$  we will have a corresponding vector of  $(Q, P)$  which we call  $\xi$ . In this notation the equivalent form for  $Q(q, p, t)$ ,  $P(q, p, t)$  is

$$\xi = \xi(\eta)$$

The Jacobian matrix  $\mathbf{M}$  of this transformation, defined by

$$M_{jk} = \frac{\partial \xi_j}{\partial \eta_k}$$

has for its inverse

$$M_{jk}^{-1} = \frac{\partial \eta_j}{\partial \xi_k}$$

and can be used to transform the symplectic form of Hamilton's equations between the two sets of variables. If the transformation is canonical then the following identities must hold:

$$\mathbf{M}\mathbf{J}\tilde{\mathbf{M}} = \mathbf{J} \quad \mathbf{J}\mathbf{M} = \tilde{\mathbf{M}}^{-1}\mathbf{J} \quad \tilde{\mathbf{M}}\mathbf{J}\mathbf{M} = \mathbf{J}$$

**Poisson brackets:**

$$[f, g] \equiv \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j}$$

or in symplectic form

$$[f, g] \equiv \frac{\partial f}{\partial \eta_j} J_{jk} \frac{\partial g}{\partial \eta_k}$$

The Poisson bracket operator is bilinear

$$[f + g, h + k] = [f, h] + [f, k] + [g, h] + [g, k]$$

antisymmetric

$$[f, g] = -[g, f]$$

and satisfies the Jacobi identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

The following identities involving Poisson brackets of the  $p$ 's and  $q$ 's are useful:

$$[\xi_j, \xi_k] = J_{jk}$$

or, written out in more detail,

$$[q_j, q_k] = [p_j, p_k] = 0 \quad ; \quad [q_j, p_k] = -[p_k, q_j] = \delta_{jk}$$

**Test for canonicity:**

New coordinates  $(Q, P)$  are canonical if and only if

$$[Q_j, P_k] = \delta_{jk}$$

For simple cases in one or two degrees of freedom this relation can sometimes be used to find  $P(q, p, t)$  if  $Q(q, p, t)$  is given.

**Motion is a canonical transformation:** The transformation of variables defined by the motion itself, i.e.,

$$Q(q_0, p_0, t) = q(q_0, p_0, t) \quad P(q_0, p_0, t) = p(q_0, p_0, t)$$

is a canonical transformation with respect to the original coordinates  $(q_0, p_0)$  of initial conditions. This means that as each little tiny step generated by small time step  $\Delta t$  changes  $(q, p)$  to new values  $q + \delta q, p + \delta p$  we may view this change as a transformation of coordinates, and if do so, the transformation is canonical.

**Time derivatives:** The time derivative of a function  $f(q, p, t)$  is given by

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}$$

Since the motion  $q(t)$  is not an explicit function of time we have

$$\dot{q} = [q, H], \quad \ddot{q} = [[q, H], H], \quad \text{etc.}$$

This allows us to write down a formal power series solution for the motion:

$$q(t) = q_0 + t[q, H]_0 + \frac{t^2}{2}[[q, H], H]_0 \dots + \frac{t^n}{n!}[[\dots[q, H]], \dots H]_0 + \dots$$

This amazing little formula is not as helpful as it looks because (i) each succeeding term in the progression of nested Poisson brackets becomes increasingly horrible unless the Hamiltonian is simple and (ii) the series often converges poorly.

**Liouville's theorem:** The flow of an incompressible fluid satisfies  $\nabla \cdot \mathbf{v} = 0$ , and for such fluid flow the density is constant along the stream lines and volumes that move with the flow are preserved. The same thing would be true in  $(q, p)$  phase space if the phase space "velocity divergence" were to vanish. This divergence is simply

$$\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} = \frac{\partial}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial H}{\partial q_k} = 0$$

so that Hamilton's equations guarantee that it is zero. Hence, phase-space volumes are preserved by the flow induced by the Hamiltonian dynamics.

This also means that if phase space is populated by lots of system points described by some initial density  $f(q, p)$ , then as the flow proceeds the density is constant along particle trajectories in phase space.

## Chapter 10: Hamilton-Jacobi Theory

### Hamilton-Jacobi Equation:

In the Hamilton-Jacobi method we seek a generating function  $S(q, Q, t)$  of type 2 which gives us a new Hamiltonian  $K = 0$ . Since we have

$$p_k = \frac{\partial S}{\partial q_k} \quad \text{and} \quad K = H(p, q, t) + \frac{\partial S}{\partial t} = 0$$

we obtain the Hamilton-Jacobi partial differential equation for  $S$ :

$$H\left(\frac{\partial S}{\partial q}, q, t\right) + \frac{\partial S}{\partial t} = 0$$

This equation is hard to solve unless  $S$  can be separated in this form:

$$S = W_1(q_1) + W_2(q_2) + \dots + T(t)$$

When the equation may be solved in this way integration constants  $P_i$  are obtained along the way as we solve for  $T(t)$  and for each of the  $W_i(q_i)$ . Each of the  $W$ -functions may depend on one or more of the constants  $Q_i$ .

A special case is when  $H$  is time-independent, in which case we choose  $T(t) = -P_1 t$  and the Hamilton-Jacobi equation is

$$H\left(\frac{\partial S}{\partial q}, q, t\right) = P_1$$

so that the integration constant  $P_1$  is the conserved energy.

(Note: every time an integration constant shows up we call it a transformed momentum. Hopefully you are used to transformations that scramble the meaning of coordinates and momenta so that this doesn't seem any weirder than

usual. Also note that I am not using Goldstein's notation in which  $\alpha$  is  $P$  and  $\beta$  is  $Q$ . These are canonical transformation variables and we ought to call a  $P$  a  $P$  and a  $Q$  a  $Q$ .)

Once  $S$  has been found we may then use the type-2 transformation equations to obtain pairs of equations which, when solved, connect  $(q, p)$  with  $(Q, P)$ :

$$p_k = \frac{\partial S}{\partial q_k} \quad ; \quad Q_k = \frac{\partial S}{\partial P_k}$$

and since the new Hamiltonian is  $K = 0$  the new coordinates  $(Q, P)$  are all constants, so  $q(Q, P, t)$  and  $p(Q, P, t)$  are the desired solution for the motion. To find the physical meaning of the  $(Q, P)$  the equations for  $(q, p)$  must be examined at  $t = 0$ , where it is found how  $(Q, P)$  relate to the initial conditions  $(q(0), p(0))$ .

I know, this sounds like gibberish, so here is a worked example: Let

$$H = \frac{p^2}{2} + q$$

The Hamiltonian is time-independent so we may write

$$S = W(q) - Pt \quad \text{and} \quad \frac{1}{2} \left( \frac{\partial W}{\partial q} \right)^2 + q = P$$

Solving for  $\partial W/\partial q$  and integrating gives

$$W(q, Q) = -\frac{2^{3/2}}{3} (P - q)^{3/2}$$

so that we have

$$S = -\frac{2^{3/2}}{3} (P - q)^{3/2} - Pt$$

This then leads to our linking equations

$$p = \frac{\partial S}{\partial q} = \sqrt{2(P - q)} \quad ; \quad Q = \frac{\partial S}{\partial P} = -t - \sqrt{2(P - q)} = -p - t$$

which can be solved to find

$$p = -Q - t \quad ; \quad q = -\frac{1}{2}Q^2 + P - Qt - \frac{1}{2}t^2$$

These are our old friends  $q = q_0 + p_0t - t^2/2$  and  $p = p_0 - t$  if we make the connections

$$Q = -p(0) \quad ; \quad P = q(0) + p(0)^2/2$$

### Action-angle variables:

The famous action-angle variables  $(w, J)$  are similar to cylindrical coordinates in phase space for particles that either *librate* (oscillate back and forth) or *rotate* (like a tether-ball at the end of its rope.) They are connected to the original variables  $(q, p)$  by a special canonical transformation that is the phase-space equivalent of cylindrical coordinates:  $J$  is like radius and  $w$  is like an angle (but doesn't go from 0 to  $2\pi$ —see below). The pendulum is the classic example of a system that does both. As long as the pendulum does not go over the top its swinging motion is called libration, but when it has enough energy to repeatedly pass over the top it is rotating. Memorize the phase portrait of the pendulum so that you have clearly in mind the difference in phase space between libration and rotation. The action  $J$  labels each orbit in phase space and the angle  $w$  is a parameter that moves the particle along the orbit. (The “angle”  $w$  is proportional to time  $t$ , but is not quite an angle: it goes from 0 to 1 over one cycle of the motion instead of from 0 to  $2\pi$  like an angle would.) The canonical coordinates  $(w, J)$  are special because for each orbit in phase space (labeled by  $J$ ) one cycle around the motion advances  $w$  by exactly 1.

The definition of the action that has this very special property is, remarkably,

$$J_k = \oint p_k dq_k$$

where  $\oint$  indicates that the integral is to be taken around one period of the librational or rotational motion. (To find  $p(q)$  to do this integral use the fact that the Hamiltonian is conserved and solve it for  $p$ .) The action  $J$  is the area enclosed

by the orbit in the  $(q, p)$  phase plane. If the dynamics problem has been solved in  $(q, p)$  space so that  $q(t)$  and  $p(t)$  are known then the integral may be performed this way:

$$J_k = \int_0^T p_k(t) \dot{q}_k(t) dt$$

Because  $p$  and  $q$  have been integrated out in the definition of  $J$  it is only a function of the Hamiltonian  $H$  (and, perhaps, the other actions or constants of the motion in multiple dimensions.) Hence, in the new canonical coordinates we have

$$H = H(J) \quad ; \quad \dot{w}_k = \frac{\partial H}{\partial J_k} \quad ; \quad \dot{J}_k = -\frac{\partial H}{\partial w_k} = 0$$

so the actions are constants and the angle variables are simply given by

$$w_k = \nu_k(J)t + w_0 \quad \text{where} \quad \nu_k(J) = \frac{\partial H}{\partial J_k}$$

where  $\nu$  is the frequency in cycles-per-second, i.e.,  $\nu = \omega/2\pi$ .

As an important example, here is the simple harmonic oscillator in action angle variables:

$$H = \frac{1}{2m}p^2 + \frac{k}{2}q^2 \quad ; \quad q = A \sin 2\pi\nu_0 t \quad , \quad p = mA2\pi\nu_0 \cos 2\pi\nu_0 t$$

$$J = \pi mA^2 \omega_0 \quad ; \quad H = \nu_0 J \quad ; \quad w = \nu_0 t$$

where

$$\nu_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$q = \sqrt{\frac{J}{\pi m \omega_0}} \sin 2\pi w \quad ; \quad p = \sqrt{\frac{mJ\omega_0}{\pi}} \cos 2\pi w$$

These new variables are especially useful for doing perturbation theory and studying chaotic dynamics.

## Chapter 11: Dynamical Chaos

Study the properties of iterated maps and understand the concepts of fixed points, stability, limit cycles, Lyapunov exponents, Poincaré maps, and the period-doubling route to chaos, especially in the logistic map.

## Chapter 12: Perturbation Theory

Perturbation theory is the art of solving hard equations iteratively when some part of the problem is small. Here's a simple example. Suppose we want to solve the equation

$$x^4 - \epsilon x^3 - 16 = 0$$

for the root near  $x = 2$  when  $\epsilon$  is small. We take advantage of the small size of  $\epsilon$  to write

$$x = (16 + \epsilon x^3)^{1/4}$$

and then build a sequence that converges to the desired answer by first ignoring the small term to find

$$x \approx 2$$

and then refining this answer by always using the most recent approximation in the small term on the right in the equation above:

$$x_{n+1} = (16 + \epsilon x_n^3)^{1/4}$$

or written out with  $\epsilon = 0.1$

$$x_0 = 2$$

$$x_1 = (16 + (.1)2^3)^{1/4} = 2.0245444689$$

$$x_2 = (16 + (.1)x_1^3)^{1/4} = 2.0254421518$$

$$x_3 = (16 + (.1)x_2^3)^{1/4} = 2.0254753765$$

$$x_4 = (16 + (.1)x_3^3)^{1/4} = 2.0254766067$$

etc. There are lots of ways to build increasingly better approximations to difficult problems, but this style of iteration will be the one we will use.

**Hamiltonian perturbation theory, iteration style:** There are many ways of doing Hamiltonian perturbation theory, including the very complex method of finding a sequence of canonical transformations that at each level makes the approximate dynamics easier to solve. Here we will study a technique very similar to the simple iteration example given above.

Suppose that we have a Hamiltonian of the form

$$H(q, p) = H_0(q, p) + H_1(q, p)$$

where  $H_1$  is small compared to  $H_0$ . We can solve for the dynamics iteratively as follows;

- (i) Solve for  $(q_0(t), p_0(t))$ , the solution when  $H = H_0$ .
- (ii) Write down the general Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H_0}{\partial p} + \frac{\partial H_1}{\partial p} \quad ; \quad \frac{dp}{dt} = -\frac{\partial H_0}{\partial q} - \frac{\partial H_1}{\partial q}$$

but in the terms in Hamilton's equations involving  $H_1$  replace the general variables  $(q, p)$  with  $(q_0(t), p_0(t))$  and solve the resulting equations to find  $(q_1(t), p_1(t))$ .

- (iii) Now repeat step (ii), but use  $(q_1, p_1)$  in the  $H_1$  terms and solve to obtain  $(q_2, p_2)$ .
- Repeat step (iii) until you either achieve the desired accuracy or you are tired.

Here is an example involving a particle falling under gravity from rest with the addition of a nonlinear, but small, upward force

$$H_0 = p^2 + q \quad ; \quad H_1 = -\epsilon q^3$$

- (i)

$$q_0 = -\frac{t^2}{2} \quad ; \quad p_0 = -t$$

(ii)

$$\frac{dq}{dt} = p \quad ; \quad \frac{dp}{dt} = -1 + 3\epsilon q^2 \approx -1 + 3\epsilon q_0^2 = -1 + \frac{3}{4}\epsilon t^4$$

which yield

$$q_1 = -\frac{t^2}{2} + \frac{1}{40}\epsilon t^6 \quad ; \quad p_1 = -t + \frac{3}{20}\epsilon t^5$$

(iii)

$$\frac{dq}{dt} = p \quad ; \quad \frac{dp}{dt} = -1 + 3\epsilon q^2 \approx -1 + 3\epsilon q_1^2 = -1 + 3\epsilon \left(-\frac{t^2}{2} + \frac{1}{40}\epsilon t^6\right)^2$$

which yield

$$q_2 = -\frac{t^2}{2} + \frac{1}{40}\epsilon t^6 - \frac{1}{1200}\epsilon^2 t^{10} + \frac{3}{291200}\epsilon^3 t^{14} \quad ; \quad p_2 = -t + \frac{3}{20}\epsilon t^5 - \frac{1}{120}\epsilon^2 t^9 + \frac{3}{20800}\epsilon^3 t^{13}$$

etc.

The pattern of things getting much worse with each step is a general pattern for Hamiltonian perturbation techniques.

**Perturbed nonlinear oscillators:** One of the most interesting uses of action-angle variables is to study what happens when a nonlinear oscillator is acted upon by a periodic perturbation. Note, however, that we can't do oscillators in the way we did the falling particle because it turns out the this iteration technique on  $(q, p)$  doesn't conserve energy. Instead we use action-angle variables. Suppose that we have a Hamiltonian already in action-angle form with a small added perturbation:

$$H = H_0(J) + H_1(J, w, t)$$

(i) To lowest order we have

$$J = J_0 = \text{constant} \quad ; \quad w = \nu(J_0)t$$

(ii) The perturbed Hamilton's equations are

$$\frac{dw}{dt} = \omega_0(J) + \frac{\partial H_1}{\partial J} \quad ; \quad \frac{dJ}{dt} = -\frac{\partial H_1}{\partial w}$$

These are hard equations to solve in general, but if we replace  $(w, J)$  in the  $H_1$  terms by  $(w_0 + w_1, J_0 + J_1)$  and Taylor expand through first order in the perturbations  $(w_1, J_1)$  they are often not so bad and progress can be made.

**Adiabatic invariance:** When a librational or rotational system is changed by allowing one of its parameters (spring constant, mass, rod length, etc.) to slowly change, then the action  $J$  is approximately conserved. The criterion for a parameter  $\lambda$  to vary slowly is that

$$\delta = \frac{T}{\lambda} \frac{d\lambda}{dt} \ll 1$$

where  $T$  is the libration or rotation period. Often the error in the adiabatic conservation of the action as  $\lambda$  is varied turns out to be of order  $\delta^2$ .

### Chapter 13: Field Theory

In field theory the role played by  $t$  in the standard Lagrangian formulation is now taken over by the vector  $x^\mu = (ct, x, y, z)$ . The generalized coordinate  $q_i(t)$  is taken over by the field quantity  $\eta(ct, x, y, z)$ , or, if there are several field quantities, (like  $E_x, E_y, E_z, B_x$ , etc.) we use the subscript  $\rho$  to denote these different fields via the notation  $\eta_\rho$ . As is customary in differential geometry, partial derivatives are indicated by commas, as in

$$\eta_{\rho,\nu} = \frac{\partial \eta_\rho}{\partial x^\nu}$$

The Lagrangian  $L(\dot{q}_i, q_i, t)$  is taken over by the Lagrangian density

$$\mathcal{L}(\eta_{\rho,\nu}, \eta_\rho, x^\nu)$$

Using this notation the Euler-Lagrange equation for the field quantities are

$$\frac{d}{dx^\nu} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}} \right) - \frac{\partial \mathcal{L}}{\partial \eta_\rho} = 0$$

and the analog of energy conservation for time-independent Lagrangians is the following conservation law for Lagrangians that don't depend on  $t$ ,  $x$ ,  $y$ , or  $z$ :

$$\frac{d}{dx^\nu} \left[ \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}} \eta_{\rho,\mu} - \mathcal{L} \delta_{\mu\nu} \right] = 0$$